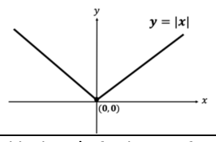
Introduction

Modulus is a function that takes a number and returns it non negative value (removes the sign of the number and returns a positive always. It simply eats the negative ☺

Example 1:	Example 2:	Example 3:	
$\|7\|$	$\|-7\|$	$\|x+2\|$	
7	The positive version of -7 is 7	Since this has a variable in and we don't' know what this is we have to take cases Plus case: $+(x+2)=x+2$ Negative case: $-(x+2)=-x-2$	You don't really need to know this, but actually a modulus is just a piecewise function $f(x)=\begin{cases}x+2, & x\ge -2\\ -x-2, & x\le -2\end{cases}$ Link this to the graph in example 1 below to understand the orange part better

Graphing Basics

The graph of $|x|$ looks like



You will probably be taught the transformation method, but the method below is far better for when the questions get harder

- **Step 1:** Find the vertex
  - ✓ x coordinate of vertex: Set what is inside the modulus equal to zero
  - ✓ y coordinate of vertex: This is the number added or subtracted outside the modulus vertex (like for a quadratic)
- **Step 2:** Take cases of the modulus for the **lines**
  - ✓ plus case (replace the modulus sign with a bracket and simplify).
  - ✓ minus case (reverse the sign of what is inside the modulus and simplify)
- **Step 3:** Find the y intercept
  - ✓ Set x = 0 (this will just be the orange point if the vertex is on the x axis)
- **Step 4:** Draw the lines (green and pink) and use the y intercept as a guid
  The positive gradient line goes upwards and the negative gradient line goes downwards. Write the equation on each line
- **Step 5:** Find the x intercept(s)
  - ✓ Set y =0 in both lines IF you can see the lines crosses the x axis (be sure to use the correct line)

Note: You can of course either use other methods. 1)transformations 2)write the function as a piecewise function and use piecewise function knowledge. However, experience tells me that students do much better with this method shown

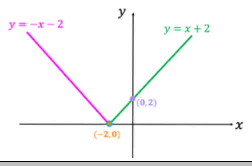
Example 1: Vertex on x axis — $y=|x+2|$

Step 1: $x+2=0$, $x=-2$. Nothing added or subtracted outside so $y=0$. Hence $(-2,0)$

Step 2:
- Plus case: $x+2$
- Negative case: $-(x+2)=-x-2$

Step 3: Let $x=0$: $y=|0+2|=2$. Hence $(0,2)$

Step 4: Let $y=0$ for both lines. $x+2=0$ gives $x=-2$; $-x-2=0$ gives $x=-2$. This is just the same as our vertex which we expect since it lies on the x axis



Insight: A modulus function is just a piecewise function $f(x)=\begin{cases}x+2, & x\ge -2\\ -x-2, & x\le -2\end{cases}$

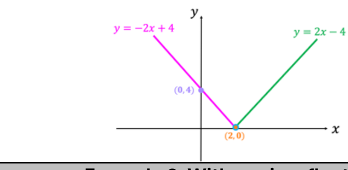
Example 2: Vertex on x axis — $y=|2x-4|$

Step 1: $2x-4=0$, $x=2$. Nothing added or subtracted outside so $y=0$. Hence $(2,0)$

Step 2:
- Plus case: $2x-4$
- Negative case: $-(2x-4)=-2x+4$

Step 3: Let $x=0$: $y=|2(0-4|=|-4|=4$. Hence $(0,4)$

Step 4: Let $y=0$ for both lines. $2x-4=0$ gives $x=2$; $-2x+4=0$ gives $x=2$. This is just the same as our vertex which we expect since it lies on the x axis



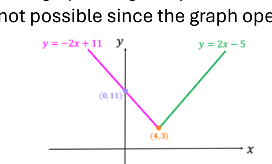
Example 3: Vertex above x axis — $y=2|x-4|+3$

Step 1: Careful here, only use the modulus part. $x-4=0$, $x=4$. 3 is added outside hence $y=3$. Hence $(4,3)$

Step 2: Careful here, consider all parts
- Plus case: $2(x-4)+3=2x-5$
- Negative case: $-2(x-4)+3=-2x+11$

Step 3: Let $x=0$: $y=2|0-4|+3=2|-4|+3=2(4)+3=11$. Hence $(0,11)$

Step 4: Let $y=0$ for both lines. $2x-5=0$ gives $x=\frac{5}{2}$; $-2x+11=0$ gives $x=\frac{11}{2}$. Watch out: The graph will guide you that these points are actually not possible since the graph opens upwards



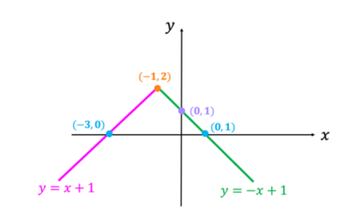
Example 4: With x axis reflection — $y=2-|x+1|$

Step 1: Careful here, only use the modulus part. $x+1=0$, $x=-1$. 2 is added outside hence $y=2$. Hence $(-1,2)$

Step 2: Careful here, only use the modulus part
- Plus case: $2-(x+1)=-x+1$
- Negative case: $2--(x+1)=x+3$

Step 3: Let $x=0$: $y=2-|0+1|=2-|1|=2-1=1$. Hence $(0,1)$

Step 4: Let $y=0$ for both lines. $-x+1=0$ gives $x=1$; $x+3=0$ gives $x=-3$



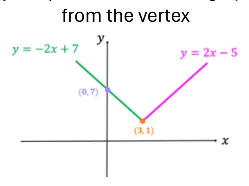
Example 3: With y axis reflection — $y=2|-x+3|+1$

Step 1: Careful here, only use the modulus part. $-x+3=0$, $x=3$. 1 is added outside hence $y=1$. Hence $(3,1)$

Step 2: Careful here, only use the modulus part
- Plus case: $2(-x+3)+1=-2x+7$
- Negative case: $-2(-x+3)+1=2x-5$

Step 3: Let $x=0$: $y=2|-0+3|+1=2|3|+1=2(3)+1=7$. Hence $(0,7)$

Step 4: Let $y=0$ for both lines. $-2x+7=0$ gives $x=\frac{7}{2}$; $2x-5=0$ gives $x=\frac{5}{2}$. Watch out: The graph will guide you that these points are actually not possible since the graph opens up from the vertex



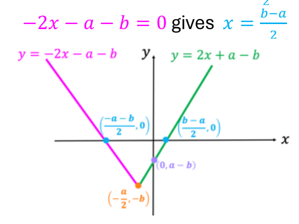
Example 3: With unknowns — $y=|2x+a|-b$, where a and b are constants and $0<a<b$

Step 1: Careful here, only use the modulus part. $2x+a=0$, $x=-\frac{a}{2}$. b is subtracted outside hence $y=b$. Hence $\left(-\frac{a}{2},b\right)$

Step 2: Careful here, only use the modulus part
- Plus case: $2x+a-b$
- Negative case: $-(2x+a)-b=-2x-a-b$

Step 3: Let $x=0$: $y=|2(0)+a|-b=|a|-b=a-b$. Hence $(0,a-b)$. Note: $a<b$ hence $a-b$ is positive

Step 4: Let $y=0$ for both lines. $2x+a-b=0$ gives $x=\frac{-a-b}{2}$; $-2x-a-b=0$ gives $x=\frac{b-a}{2}$



Transformations

y transformations – outside the modulus (these occur outside the modulus)	x transformations (these occur inside the modulus - do the opposite to what you expect)
$a\|f(x)\|\pm d$ • Apply modulus • Multiply y by a • Add/Subtract d to/from y	$f(\|bx\pm c\|)$ • Divide x by b • Subtract/add c from/to y • Apply modulus

Given $f(x)$ notation

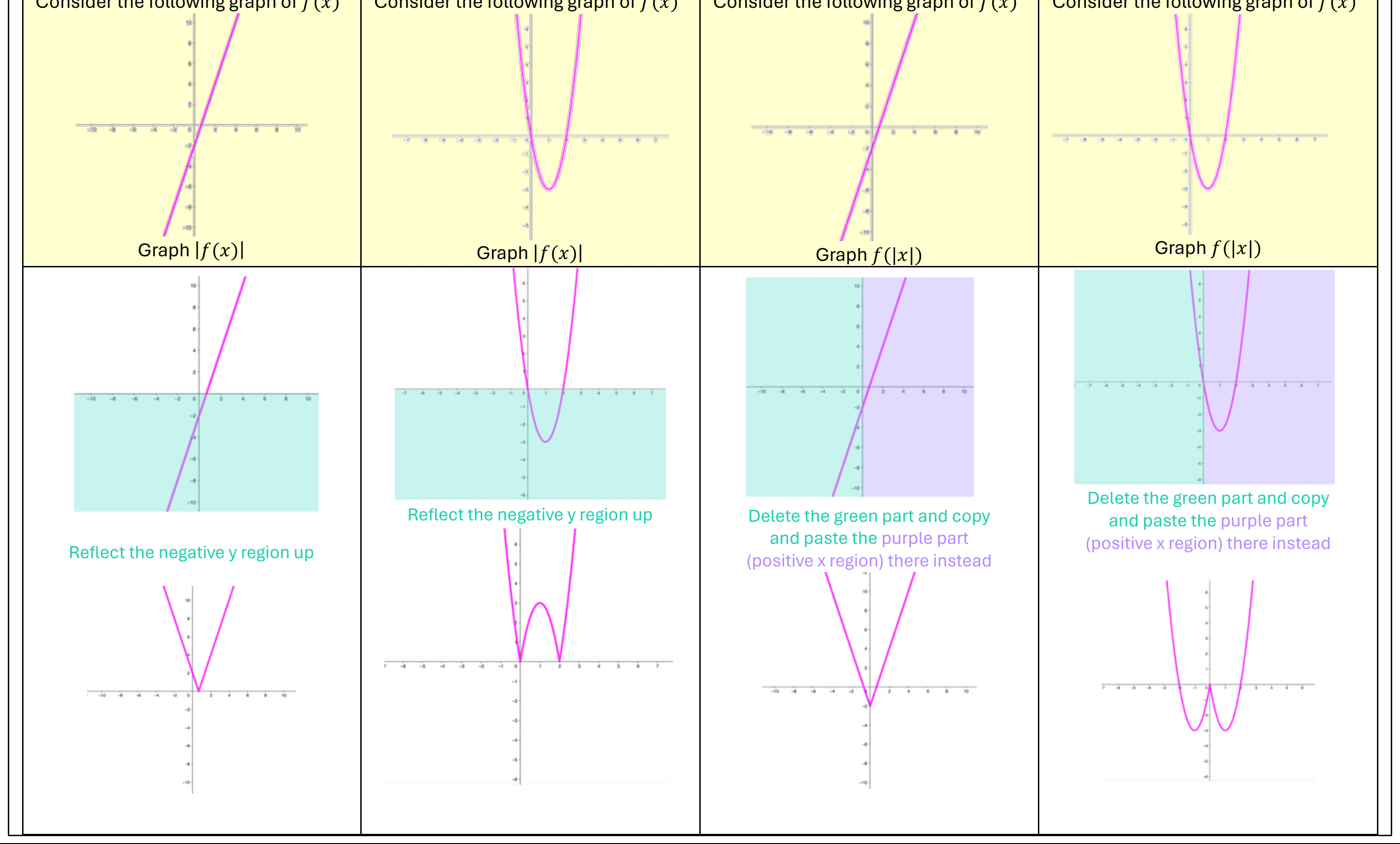
Example 1: 2025 June Pure Paper 2 Q1	Example 2
The point $P(6,-4)$ lies on the curve with equation, $y=f(x), x\in\mathbb{R}$. Find the point to which P is mapped when the curve with equation $y=f(x)$ is transformed to the curve with equation $y=2\|f(x)\|-3$	The point $P(-6,-4)$ lies on the curve with equation, $y=f(x), x\in\mathbb{R}$. Find the point to which P is mapped when the curve with equation $y=f(x)$ is transformed to the curve with equation $y=f(2\|x\|-3)$
These are all y transformations (we apply the modulus first) • Apply modulus • Multiply by 2 • Subtract 3 $(6,-4)\to(6,4)\to(6,8)\to(6,5)$	These are all x transformations (we apply the modulus last) • Add 3 • Divide by 2 • Apply modulus $(-6,-4)\to(-9,-4)\to\left(-\frac{9}{2},-4\right)\to\left(\frac{9}{2},-4\right)$

Given $f(x)$ graph

Given an $f(x)$ graph, you need to know how to draw $|f(x)|$ and $f(|x|)$ graphs

- $|f(x)|$ reflect any negative y values up to the positive y axis. Why? Since y can never be negative so we reflect negative values up
- $f(|x|)$ delete everything to the left of the line $x=0$ and copy and paste anything to the right of the line onto the left of the line $x=0$. Why? Since negative values of x become positive when taken the modulus so negative values of x never get seen, they just revert to the positive version
- $f(|x-a|)$ delete everything to the left of the line $x=a$ i.e. where $x-a$ is positive hence $x>a$ and copy and paste anything to the right of this line onto the left of this line. **Careful** though as both people do not realise this last bullet point as it very rarely comes up and hence teachers usually do not mentioned it.

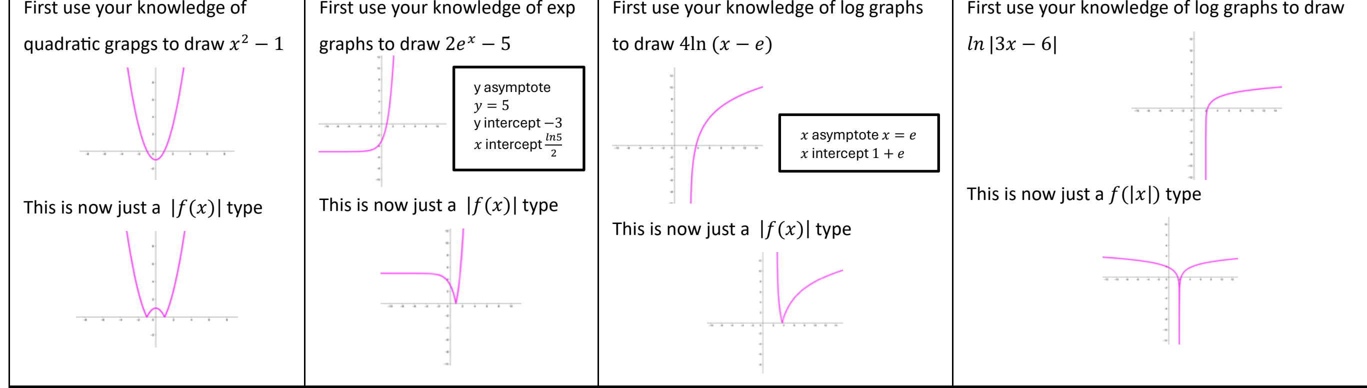
Example 1	Example 2	Example 3	Example 4
Consider the following graph of $f(x)$	Consider the following graph of $f(x)$	Consider the following graph of $f(x)$	Consider the following graph of $f(x)$
Graph $\|f(x)\|$	Graph $\|f(x)\|$	Graph $f(\|x\|)$	Graph $f(\|x\|)$
Reflect the negative y region up	Reflect the negative y region up	Delete the green part and copy and paste the purple part (positive x region) there instead	Delete the green part and copy and paste the purple part (positive x region) there instead



Graphing Harder Functions – Quadratics, Logs and Exponentials

See my how to graph cheat sheet first if you're not comfortable with graphing all types of functions.

Example 1: Quadratic	Example 2: Exponentials	Example 3: Quadratic	Example 4: Logarithmic
Sketch $y=\|x^2-1\|$	Sketch $\|2e^x-5\|$	$y=\|4\ln(x-e)\|$	Sketch $\ln\|3x-6\|$
First use your knowledge of quadratic grapgs to draw x^2-1 This is now just a $\|f(x)\|$ type	First use your knowledge of exp graphs to draw $2e^x-5$ This is now just a $\|f(x)\|$ type	First use your knowledge of log graphs to draw $4\ln(x-e)$ This is now just a $\|f(x)\|$ type	First use your knowledge of log graphs to draw $\ln\|3x-6\|$ This is now just a $f(\|x\|)$ type



Solving Equalities

Important tips: There are 3 ways to deal with these

Way 1: Replace the modulus with the plus and minus case and solve (check your answers at the end in case some need to be ignored)

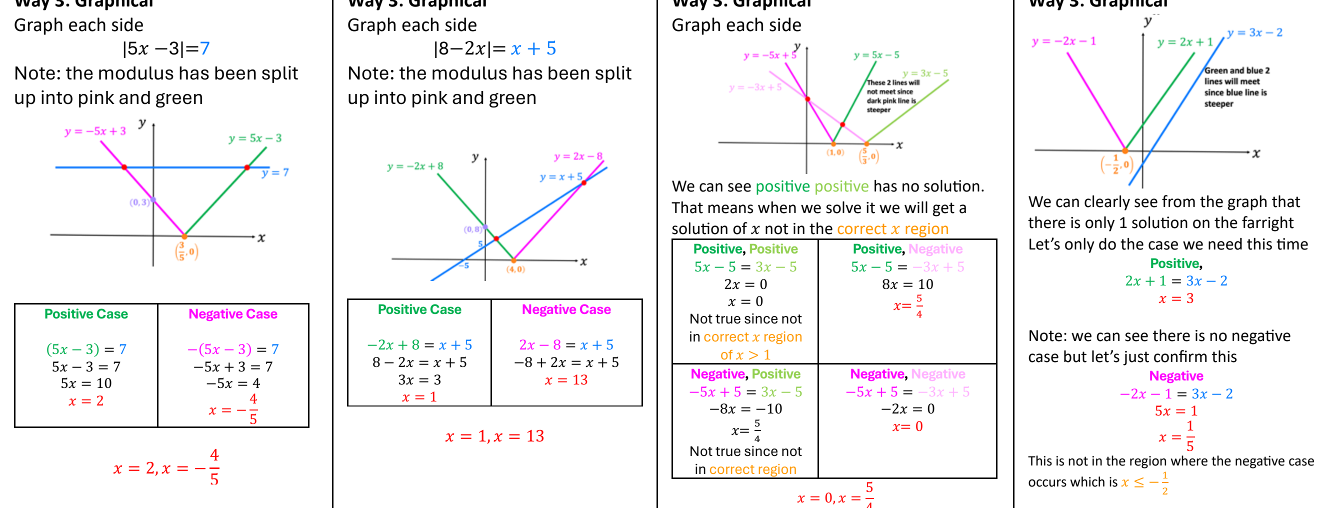
Way 2: Square both sides and solve (check your answers at the end in case some need to be ignored)

Way 3: graph both sides of the equation. This is a bit harder, but it becomes important for when you solve inequalities later so learn it.

Tip 1: If two modulus graphs on the same axis, have the line with a steeper gradient looking steeper

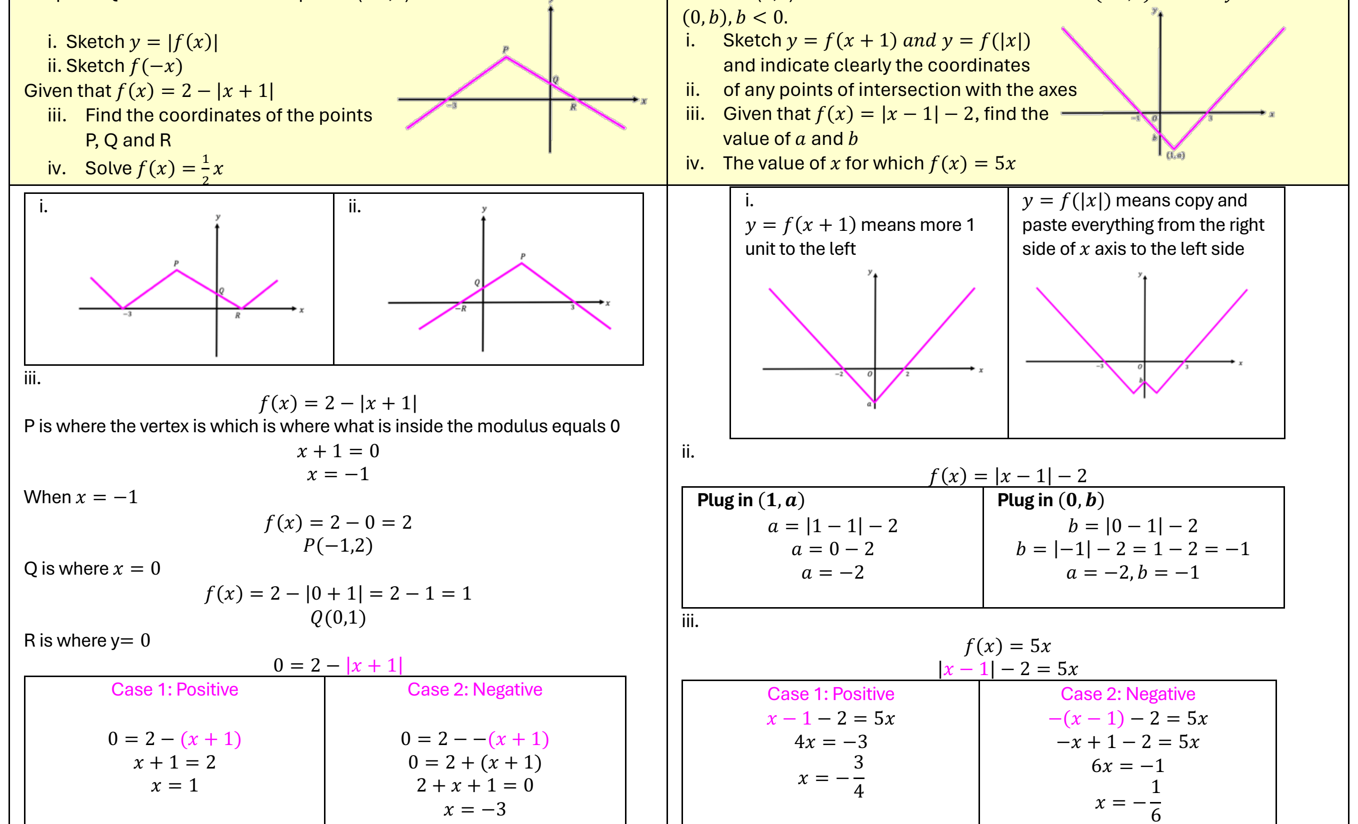
Tip 2: Make sure the solution you get makes sense in terms of the x axis region the line relates to. If not, we ignore it.

Example 1: Modulus on one side	Example 2: Modulus on one side	Example 3: Modulus on both sides	IMPORTANT EXAMPLE Example 4: Ignoring Solutions that are not valid
Solve $\|5x-3\|=7$	Solve $\|8-2x\|=x+5$	Solve $\|5x-3\|=\|3x-5\|$	Solve $\|2x+1\|=3x-2$
Way 1: Take cases Positive case: $5x-3=7$, $x=2$; Negative case: $-(5x-3)=7$, $x=-\frac{4}{5}$. Check with original question: When $x=2$: $\|5(2)-3\|=7$, $7=7$ ∴ true since true statement. When $x=-\frac{4}{5}$: $\|5(-\frac{4}{5})-3\|=7$, $7=7$ ∴ true since true statement	**Way 1: Take cases** Positive case: $8-2x=x+5$, $3x=3$, $x=1$; Negative case: $-(8-2x)=x+5$, $-8+2x=x+5$, $x=13$. Check with original question: When $x=1$: $\|8-2(1)\|=1+5$, $6=6$ ∴ true since true statement. When $x=13$: $\|8-2(13)\|=13+5$, $18=18$ ∴ true since true statement	**Way 1: Take cases** Positive, Positive: $5x-5=3x-5$, $2x=0$, $x=0$. Positive, Negative: $5x-5=-(3x-5)$, $8x=10$, $x=\frac{5}{4}$. Negative, Positive: $-(5x-5)=3x-5$, $-8x=-10$, $x=\frac{5}{4}$. Negative, Negative: $-(5x-5)=-(3x-5)$, $-2x=0$, $x=0$. Check with original question: When $x=0$: $\|5(0)-5\|=\|3(0)-5\|$, $5=5$ ∴ true since true statement. When $x=\frac{5}{4}$: $\|5(\frac{5}{4})-5\|=\|3(\frac{5}{4})-5\|$, $1.25=1.25$ ∴ true since true statement	**Way 1: Take cases** Positive case: $2x+1=3x-2$, $x=3$; Negative case: $-(2x+1)=3x-2$, $5x=1$, $x=\frac{1}{5}$. Check with original question: When $x=3$: $\|2(3)+1\|=3(3)-2$, $7=7$ ∴ true since true statement. When $x=\frac{1}{5}$: $\|2(\frac{1}{5})+1\|=3(\frac{1}{5})-2$, $1.4\ne-1.4$ ∴ NOT TRUE. $x=3$ is the only solution
Way 2: Square both sides $(5x-3)^2=7^2$, $25x^2-30x+9=49$, $25x^2-30x-40=0$, $x=2, x=-\frac{4}{5}$. Check like in way 1 above	**Way 2: Square both sides** $(8-2x)^2=(x+5)^2$, $64-32x+4x^2=x^2+10x+25$, $3x^2-42x+39=0$, $x=1, x=13$. Check like in way 1 above	**Way 2: Square both sides** $(5x-5)^2=(3x-5)^2$, $5x^2-50x+25=9x^2-30x+225$, $4x^2+20x+200=0$, $x=0, x=\frac{5}{4}$. Check like in way 1 above	**Way 2: Square both sides** $(2x+1)^2=(3x-2)^2$, $4x^2+4x+4=9x^2-12x+4$, $5x^2-20x=0$, $x=\frac{1}{5}, x=3$. Check like in way 1 above (ignore $x=\frac{1}{5}$)
Way 3: Graphical Graph each side $\|5x-3\|=7$. Note: the modulus has been split up into pink and green. Positive Case: $(5x-3)=7$, $5x=10$, $x=2$. Negative Case: $-(5x-3)=7$, $-5x+3=7$, $-5x=4$, $x=-\frac{4}{5}$. $x=2, x=-\frac{4}{5}$	**Way 3: Graphical** Graph each side $\|8-2x\|=x+5$. Note: the modulus has been split up into pink and green. Positive Case: $-2x+8=x+5$, $8-2x=x+5$, $3x=3$, $x=1$. Negative Case: $2x-8=x+5$, $-8+2x=x+5$, $x=13$. $x=1, x=13$	**Way 3: Graphical** Graph each side. We can see positive positive has no solution. That means when we solve it we will get a solution of x not in the correct region. Positive, Positive: $5x-5=3x-5$, $2x=0$, $x=0$, Not true since not in correct region. Positive, Negative: $5x-5=-3x+5$, $8x=10$, $x=\frac{5}{4}$. Negative, Positive: $-5x+5=3x-5$, $-8x=-10$, $x=\frac{5}{4}$, Not true since not in correct region. Negative, Negative: $-5x+5=-3x+5$, $-2x=0$, $x=0$. $x=0, x=\frac{5}{4}$	**Way 3: Graphical** We can clearly see from the graph that there is only 1 solution on the farright. Let's only do the case we need this time. Positive: $2x+1=3x-2$, $x=3$. Note: we can see there is no negative case but let's just confirm this. Negative: $-2x-1=3x-2$, $5x=1$, $x=\frac{1}{5}$. This is not in the region where the negative case occurs which is $x\le-\frac{1}{2}$



Example 7: June 2025 Paper 2 Q12	Example 4:	Example 5	Example 6
$f(x)=4\|x-3\|-5$. Given that a is a constant and $\|a\|=1$. Find the possible values of $f(a)$	$f(x)=\|x\|, g(x)=3x+5$. i. Solve $g(x+2)=f(-12)$ ii. Determine the values of x for which $x+f(x)=0$	$f(x)=\|2x+a\|+3a$, $g(x)=5x-4a$ where a is a positive constant. Solve for x the equation $gf(x)=31a$	Given that $\|x+3a\|=5a$, where a is a positive constant. Find, in terms of a, the possible values of $\|x+7a\|-\|x-7a\|$
$\|a\|=1$ means $a=1$ or $a=-1$. $f(1)=4\|1-3\|-5=3$, $f(-1)=4\|-1-3\|-5=11$. 3 and 11	i. $g(x+2)=3(x+2)+5=3x+11$; $g(x+2)=f(-12)$, $3x+11=\|-12\|$, $3x+11=12$, $x=\frac{1}{3}$. ii. $x+\|x\|=0$, $x=-\|x\|$. $x\le0$	$f(x)=\|2x+a\|+3a$, $g(x)=5x-4a$. $g(f(x))=31a$, $5(\|2x+a\|+3a)-4a=31a$, $5\|2x+a\|+3a\|-4a=31a$. Case 1: Positive Case: $5(2x+4a)-4a=31a$, $10x+20a-4a=31a$, $10x+16a=31a$, $10x=15a$, $x=\frac{15a}{10}=\frac{3a}{2}$. Case 2: Negative Case: $5(-2x+2a)-4a=31a$, $-10x+10a=31a$, $-10x+6a=31a$, $10x=-25a$, $x=-\frac{5a}{2}$. $x=\frac{3a}{2}$ or $-\frac{5a}{2}$	$\|x+3a\|=5a$: $x+3a=5a$, $x=2a$; $-(x+3a)=5a$, $-x-3=5a$, $x=-8a$. Now solve for $\|x+7a\|-\|x-7a\|$. When $x=2a$: $\|2a+7a\|-\|2a-7a\|=9a-\|-5a\|=9a-5a=4a$. When $x=-8a$: $\|-8a+7a\|-\|-8a-7a\|=\|-a\|-\|-15a\|=a-15a=-14a$. Hence possible values are $4a$ and $-14a$.

Example 7: C3 June 2008 Q3	Example 8: C3 June 2005 Q 6
The diagram shows the graph of $y=f(x), x\in\mathbb{R}$. The graph consists of two line segments that meet at the point P. The graph cuts the y axis at the point Q and the x axis at the points (−3,0) and R. i. Sketch $y=\|f(x)\|$ ii. Sketch $f(-x)$ Given that $f(x)=2-\|x+1\|$ iii. Find the coordinates of the points P, Q and R iv. Solve $f(x)=\frac{1}{2}x$	The diagram shows part of the graph of $y=f(x), x\in\mathbb{R}$. The graph consists of two line segments that meet at the point $(1,a)$, $a<0$. One line meets the x axis at (3,0). The other line meets the x axis at (−1,0) and the y axis at $(0,b)$, $b<0$. i. Sketch $y=f(x+1)$ and $y=f(\|x\|)$ and indicate clearly the coordinates of any points of intersection with the axes ii. Given that $f(x)=\|x-1\|-2$, find the value of a and b iv. The value of x for which $f(x)=5x$
iii. $f(x)=2-\|x+1\|$. P is where the vertex is which is where what is inside the modulus equals 0: $x+1=0$, $x=-1$. When $x=-1$: $f(x)=2-0=2$, $P(-1,2)$. Q is where $x=0$: $f(x)=2-\|0+1\|=2-1=1$, $Q(0,1)$. R is where $y=0$: $0=2-\|x+1\|$. Case 1: Positive: $0=2-(x+1)$, $x+1=2$, $x=1$. Case 2: Negative: $0=2--(x+1)$, $0=2+(x+1)$, $2+x+1=0$, $x=-3$. We already have this point shown on the graph. $R(1,0)$ iv. $2-\|x+1\|=\frac{1}{2}x$. Case 1: Positive: $2-(x+1)=\frac{1}{2}x$, $2-x-1=\frac{1}{2}x$, $x=\frac{2}{3}$. Case 2: Negative: $2--(x+1)=\frac{1}{2}x$, $2+x+1=\frac{1}{2}x$, $x=-6$. $x=-6, x=\frac{2}{3}$ Note: As usual check with a graph if both of these are valid or just plug back both into the original equation $2-\|x+1\|=\frac{1}{2}x$ and see if both give true statements. They do, hence both valid.	i. $y=f(x+1)$ means move 1 unit to the left. $y=f(\|x\|)$ means copy and paste everything from the right side of x axis to the left side ii. $f(x)=\|x-1\|-2$. Plug in $(1,a)$: $a=\|1-1\|-2$, $a=0-2$, $a=-2$. Plug in $(0,b)$: $b=\|0-1\|-2$, $b=\|-1\|-2=1-2=-1$, $a=-2, b=-1$ iii. $f(x)=5x$, $\|x-1\|-2=5x$. Case 1: Positive: $x-1-2=5x$, $4x=-3$, $x=-\frac{3}{4}$. Case 2: Negative: $-(x-1)-2=5x$, $-x+1-2=5x$, $6x=-1$, $x=-\frac{1}{6}$ Note: As usual check with a graph if both of these are valid or just plug back both into the original equation $\|x-1\|-2=5x$: $\|-\frac{3}{4}-1\|-2=5\left(-\frac{3}{4}\right)$ and $\|-\frac{1}{6}-1\|-2=5\left(-\frac{1}{6}\right)$. $\frac{7}{4}-2\ne-\frac{15}{4}$ and $\frac{7}{6}-2=-\frac{5}{6}$. Hence $x=-\frac{1}{6}$ is the only solution

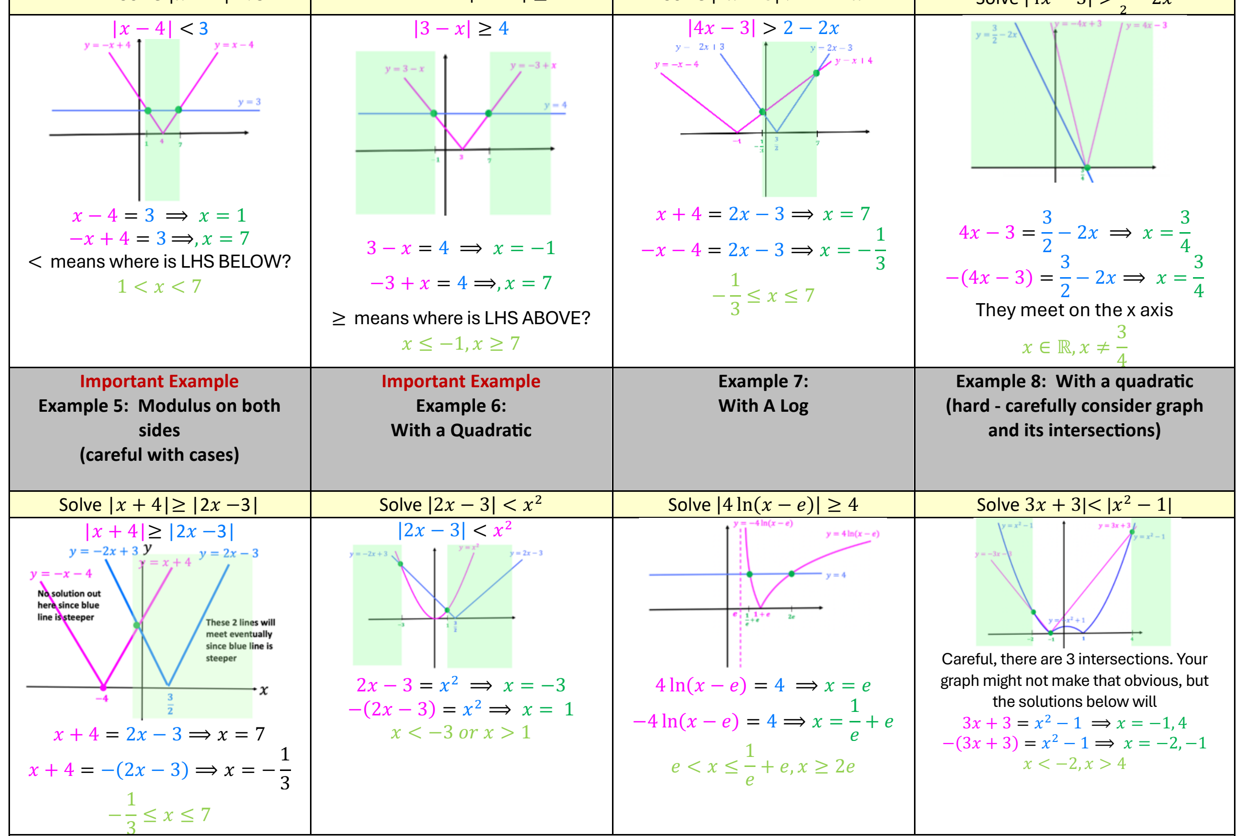


Solving Inequalities

Watch out of these! Here you **NEED** to **GRAPH** and do one more step than with equalities. After finding the solutions, check where the left hand side graph is above (if $>$ or $\ge$) or below (if $<$ or $\le$) the right hand side graph

Example 1: Modulus on one side	Example 2: Modulus on one side	Example 3: Modulus on one side	Example 4: Modulus on one side (unusual solution)
Solve $\|x-4\|<3$	Solve $\|3-x\|\ge4$	Solve $\|4x-3\|>2-2x$	Solve $\|4x-3\|>\frac{3}{2}-2x$
$x-4=3\Rightarrow x=1$ $-x+4=3\Rightarrow x=7$ $<$ means where is LHS BELOW? $1<x<7$	$3-x=4\Rightarrow x=-1$ $-3+x=4\Rightarrow x=7$ $\ge$ means where is LHS ABOVE? $x\le-1, x\ge7$	$x+4=2x-3\Rightarrow x=7$ $-x-4=2x-3\Rightarrow x=-\frac{1}{3}$ $-\frac{1}{3}\le x\le7$	$4x-3=\frac{3}{2}-2x\Rightarrow x=\frac{3}{4}$ $-(4x-3)=\frac{3}{2}-2x\Rightarrow x=\frac{3}{4}$ They meet on the x axis $x\in\mathbb{R}, x\ne\frac{3}{4}$

Important Example Example 5: Modulus on both sides (careful with cases)	Important Example Example 6: With a Quadratic	Example 7: With A Log	Example 8: With a quadratic (hard - carefully consider graph and its intersections)
Solve $\|x+4\|\ge\|2x-3\|$	Solve $\|2x-3\|<x^2$	Solve $\|4\ln(x-e)\|\ge4$	Solve $3x+3\|<\|x^2-1\|$
No solution out here since blue line is steeper. These 2 lines will meet eventually since blue line is steeper $x+4=2x-3\Rightarrow x=7$ $x+4=-(2x-3)\Rightarrow x=-\frac{1}{3}$ $-\frac{1}{3}\le x\le7$	$2x-3=x^2\Rightarrow x=-3$ $-(2x-3)=x^2\Rightarrow x=1$ $x<-3$ or $x>1$	$4\ln(x-e)=4\Rightarrow x=e$ $-4\ln(x-e)=4\Rightarrow x=\frac{1}{e}+e$ $e<x\le\frac{1}{e}+e, x\ge2e$	Careful, there are 3 intersections. Your graph might not make that obvious, but the solutions below will $3x+3=x^2-1\Rightarrow x=-1,4$ $-(3x+3)=x^2-1\Rightarrow x=-2,-1$ $x<-2, x>4$



Example 9 (Between)

Solve $1\le|x-4|\le3$

We can separate it into two inequalities:

$|x-4|\ge1$ and $|x-4|\le3$

The results we got are $x\le3, x\ge5$, $1\le x\le7$. Now we take the overlap which is

$1\le x\le3, 5\le x\le7$

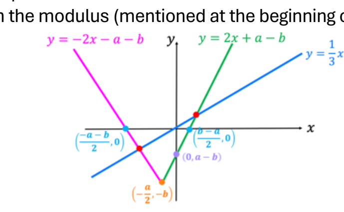
Hard Exam Questions

Example 1

a and b are constants and $0<a<b$.
Solve for x in the equation $|2x+a|-b=\frac{1}{3}x$, giving your answer in terms of a and b

$|2x+a|-b=\frac{1}{3}x$

We need to graph both sides. For the left hand side use our knowledge of how the graph the modulus (mentioned at the beginning of this document).



Note: we don't need the x axis intersections to answer this question, but let's include them anyway

We now need to find the intersection points

$|2x+a|-b=\frac{1}{3}x$

Remember that we take cases for the modulus (plus and minus case)

$(2x+a)-b=\frac{1}{3}x$ and $-2x-a-b=\frac{1}{3}x$

Positive case	Negative case
$(2x+a)-b=\frac{1}{3}x$ $2x-\frac{1}{3}x=b-a$ $\frac{5}{3}x=b-a$ $x=\frac{3}{5}(b-a)$	$-2x-a-b-=\frac{1}{3}x$ $-2x-\frac{1}{3}x=a+b$ $-\frac{7}{3}x=a+b$ $x=-\frac{3}{7}(a+b)$

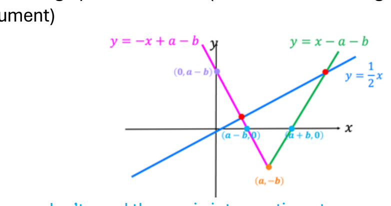
$x=\frac{3}{5}(b-a), x=-\frac{3}{7}(a+b)$

Example 2

$y=|x-a|-b$ where a and b are positive constants and a$>b>$ 0. Find the complete set of values of x for which $|x-a|-b<\frac{1}{2}x$. Giving your answer for x a in terms of b

$|x-a|-b=\frac{1}{2}x$

We need to graph both sides. For the left hand side use our knowledge of how the graph the modulus (mentioned at the beginning of this document)



Note: we don't need the x axis intersections to answer this question, but let's include them anyway

We find the intersection points by solving as an equality (forget about te inequality for now)

$|x-a|-b=\frac{1}{2}x$

Remember that we take cases for the modulus (plus and minus case)

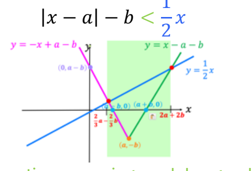
$(x-a)-b=\frac{1}{2}x$ and $-(x-a)-b=\frac{1}{2}x$

Positive case	Negative case
$(x-a)-b=\frac{1}{2}x$ $x-a-b=\frac{1}{2}x$ $x-\frac{1}{2}x=a+b$ $\frac{1}{2}x=a+b$ $x=2(a+b)$ $x=2a+2b$	$-(x-a)-b=\frac{1}{2}x$ $-x+a-b=\frac{1}{2}x$ $-x-\frac{1}{2}x=-a+b$ $-\frac{3}{2}x=-a+b$ $x=-\frac{2}{3}(-a+b)$ $x=\frac{2}{3}a-\frac{2}{3}b$

So, we know we intersect twice. Let's put all this on a graph

We need to solve the inequality part

$|x-a|-b<\frac{1}{2}x$



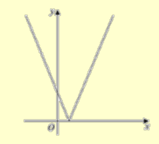
We are solving a $<$ question, meaning modulus graph must be BELOW

He $\frac{2}{3}a-\frac{2}{3}b<x<2a+2b$

Hardest Exam Questions

Example 1: October 2021 Pure 2 Q11

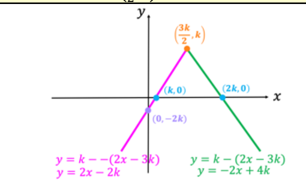
The diagram shows a sketch of the graph with equation y = $|2x-3k|$, where k is a positive constant.



i. Sketch the graph with equation $y=f(x)$ where $f(x)=k-|2x-3k|$ Stating
- the coordinates of the maximum point
- the coordinates of any points where the graph cuts the coordinate axes

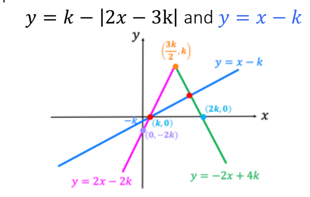
ii. Find, in terms of k, the set of values of x for which $k-|2x-3k|>x-k$, giving your answer in set notation

iii. Find in terms of k, the coordinates of the minimum point of the graph with equation $y=3-5f\left(\frac{1}{2}x\right)$



ii. The LHS of the equation is already graphed. Let's add the graph the RHS of the equation to the graph.

$y=k-|2x-3k|$ and $y=x-k$

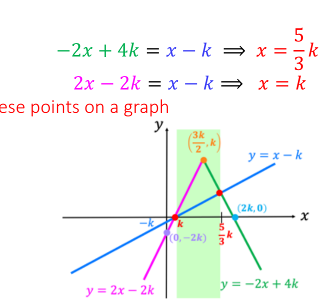


We need the intersection points. Don't worry if your graph isn't too accurate or you don't know where the intersection points are. The algebra below will tell.

$-2x+4k=x-k\Rightarrow x=\frac{5}{3}k$

$2x-2k=x-k\Rightarrow x=k$

Let's indicate these points on a graph



$k-|2x-3k|>x-k$

We are solving a question says $>$, meaning modulus graph must be ABOVE

Hence for what x values is the pink graph (LHS) ABOVE the blue graph (RHS). i.e. in the green shaded region

$k<x<\frac{5k}{3}$

iii.

Coordinates of maximum point for $f(x)$ is $\left(\frac{3k}{2},k\right)$. We have $3-5f\left(\frac{1}{2}x\right)$

x is multiplied by 2: $\frac{3k}{2}\times2=3k$

The graph is inverted: $k=-k$

And stretched by 5: $-k\times5=-5k$

And moved up by 3: $3-5k$

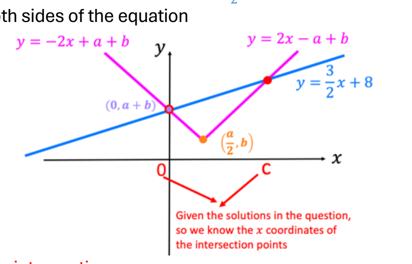
The minimum point is $(3k,3-5k)$

Example 2: C3 June 2017 Q6

Given that the equation $|2x-a|+b=\frac{3}{2}x+8$, where a and b are positive constants, has a solution at x=0 and $x=c$. Find c in terms of a

$|2x-a|+b=\frac{3}{2}x+8$

Graph both sides of the equation



To find the intersections:

We find the intersection points by solving as an equality

$|2x-a|+b=\frac{3}{2}x+8$

Remember that we take cases for the modulus (plus and minus case)

$(2x-a)+b=\frac{3}{2}x+8$ and $-(2x-a)+b=\frac{3}{2}x+8$

We solve each

$(2x-a)+b=\frac{3}{2}x+8$ $2x-a+b=\frac{3}{2}x+8$ Re-arrange for x $2x-\frac{3}{2}x=a-b+8$ $\frac{1}{2}x=a-b+8$ $x=2(8+a-b)$ Given this case occurs when $x=c$ $c=2(8+a-b)$ $c=16+2a-2b$	$-(2x-a)+b=\frac{3}{2}x+8$ $-2x+a+b=\frac{3}{2}x+8$ Re-arrange for x $-2x-\frac{3}{2}x=-a-b+8$ $-\frac{7}{2}x=-a-b+8$ $x=\frac{2}{7}(a+b-8)$ Given this case occurs when $x=0$ $0=\frac{2}{7}(a+b-8)$ $a+b-8=0$ $a+b=8$ $b=8-a$

Let's solve the last 2 equations simultaneously to get c in terms of a as asked in the question

$c=16+2a-2(8-a)$

$c=16+2a-16+2a$

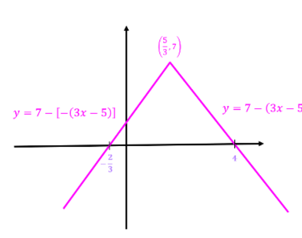
$c=4a$

Geometry Style

Number of solutions

The equation $7-|3x-5|=k$, where k is a constant has 2 distinct real solutions.
Write down the range of possible values of k

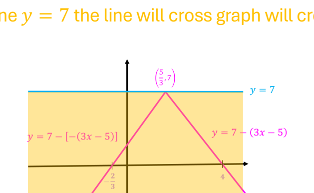
Way 1: Geometrical



This means if we draw a horizontal line ($y=k$ is just a horizontal line) it should touch the graph in 2 places only

At $y=7$ the line will only cross the graph once (draw line $y=7$)

Everywhere below the line $y=7$ the line will cross graph will cross twice (at 2 different points)



So, the range of values of k is

$$k<7$$

Way 2: Algebraic (use discriminant)

$$7-|3x-5|=k$$
$$7-k=|3x-5|$$

Square each side so it is always positive

$$(7-k)^2=|3x-5|^2$$
$$49-14k+k^2=9x^2-30x+25$$
$$0=9x^2-30x+25-49+14k-k^2$$
$$0=9x^2-30x-24+14k-k^2$$

Find the discriminant. As there are 2 distinct real solutions, $b^2-4ac>0$

$$0=9x^2-30x+(-24+14k-k^2)$$
$$(-30)^2-4(9)(-24+14k-k^2)>0$$
$$900-36(-24+14k-k^2)>0$$
$$900+864-504k+36k^2>0$$
$$36k^2-504k+1794>0$$
$$(k-7)^2>0$$
$$k<7$$

Sample Pure 2 Q11
With a restricted domain

$$f(x)=2|3-x|+5, x\geq 0$$

i. State the range of f

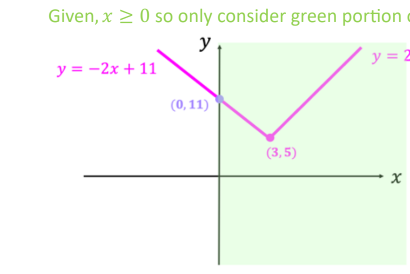
ii. Solve the equation $f(x)=\frac{1}{2}x+30$

Given that the equation $f(x)=k$, where k is a constant has two distinct roots.

iii. State the set of possible values of k

i.

Given, $x\geq 0$ so only consider green portion of graph



$$f(x)\geq 5$$

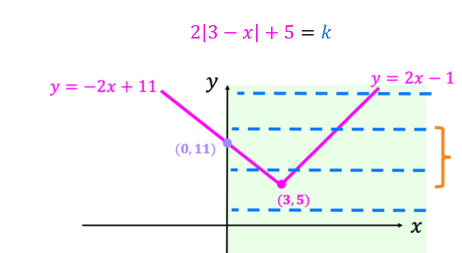
Note: See domain and range cheat sheet to properly understand this

ii.

$$x=\frac{62}{3}, x\neq -\frac{38}{5}$$

iii.

$$2|3-x|+5=k$$



when $x=0, y=11$

$$5<k\leq 11$$

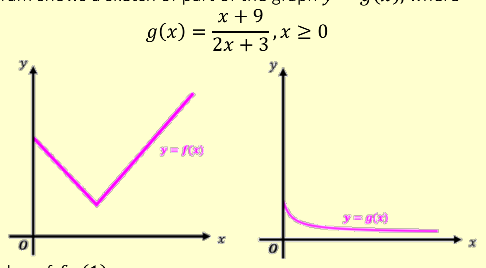
C34 Jan 2014 IAL Q 4 – Modulus

The left diagram shows a sketch of part of the graph $y=f(x)$, where

$$f(x)=2|3-x|+5, x\geq 0.$$

The right diagram shows a sketch of part of the graph $y=g(x)$, where

$$g(x)=\frac{x+9}{2x+3}, x\geq 0$$



i. Find the value of $fg(1)$
ii. State the range of g
iii. Find $g^{-1}(x)$ and state the domain

Given that the equation $f(x)=k$, where k is a constant, has exactly two roots

iv. State the range of possible values of k

$$f(x)=2|3-x|+5, g(x)=\frac{x+9}{2x+3}$$

i.

$$fg(1)$$
$$g(1)=\frac{1+9}{2(1)+3}$$
$$g(1)=2$$
$$f(2)=2|3-2|+5=2+5=7$$

ii.

$$g(x)=\frac{x+9}{2x+3}$$

When $x=0$:

$$g(x)=\frac{9}{3}=3$$

We need to see how low the graph tends to on the right - hand side

To do this we can find the inverse and then set the denominator equal to zero or we can just analyse the function as $x\to\infty$. The function tends to 0.5 but doesn't touch it.

$$0.5<g(x)\leq 3$$

iii

$$y=\frac{x+9}{2x+3}$$
$$x=\frac{y+9}{2y+3}$$
$$x(2y+3)=y+9$$
$$2xy+3x=y+9$$
$$2xy-y=9-3x$$
$$y(2x-1)=9-3x$$
$$y=\frac{9-3x}{2x-1}$$
$$2x-1=0$$
$$x=\frac{1}{2}$$

The domain of the inverse is the range of $g(x)$, but use the letter x

$$0.5<x\leq 3$$

iv.

$$y=2|3-x|+5$$

When $x=0$:

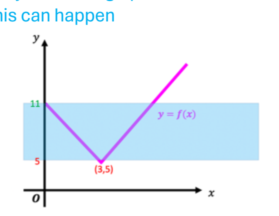
$$y=2|3-0|+5=11$$

Let's find the vertex point (we set what is inside the modulus equal to zero to find the x coordinate)

$$3-x=0$$
$$x=3$$

When $x=3$: $y=\ y=2(0)+5=5$

$2|3-x|+5=k$ having 2 roots means we can draw a vertical k on the graph of $2|3-x|+5$ and it must only cross the graph twice. The blue rectangle demonstrates the values of y for which this can happen



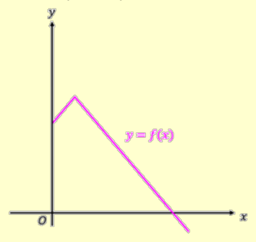
$$5<k\leq 11$$

Note: Notice how we do not equal 5 since if we touch the vertex we only intersect the graph ones

A Level October 2020 IAL P3 Q4

The diagram below shows a sketch of part of the graph with equation $y=f(x)$ where

$$f(x)=21-2|2-x|, x\geq 0$$



i. Find $ff(6)$

ii. Solve the equation $f(x)=5x$

Given that the equation $f(x)=k$, where k is a constant, has exactly two roots,

iii. State the set of possible values of k

The graph with equation $y=f(x)$ is transformed onto the graph with equation

$$y=af(x-b).$$

The vertex of the graph with equation $y=af(x-b)$ is $(6,3)$

Given that a and b are constants,

iv. Find the value of a and the value of b

$$f(x)=21-2|2-x|$$
$$f(6)=21-2|2-6|=21-2(4)=13$$
$$f(13)=21-2|2-15|=21-2(13)=-5$$

$f(x)=5x$

$$21-2|2-x|=5x$$
$$21-2|2-x|=5x$$

Remember that we take cases for the modulus (plus and minus case)

We solve each case

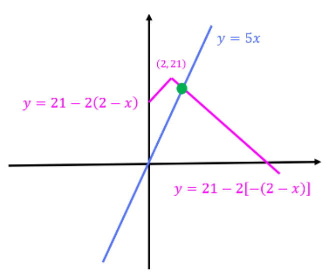
Positive case:	Negative case:
$21-2(2-x)=5x$	$21-2[-(2-x)]=5x$
$21-4+2x=5x$	$21+4-2x=5x$
$3x=17$	$7x=25$
$x=\frac{17}{3}$	$x=\frac{25}{7}$
If you look at the graph below you can see this is not possible since the positive case is only for x values smaller than -4	

Let's graph just to make sure that these are both valid solutions

$$21-2|2-x|=5x$$

Positive case: $y=21-2(2-x)=21-4+2x=2x+19$
Negative case: $y=21-2[-(2-x)]=21+4+2x=-2x+25$

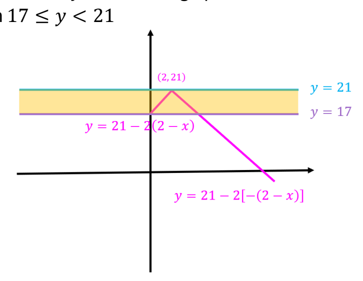
When $x=2, y=21$



we can see the only solution is for the negative case

$$x=\frac{25}{7}$$

Given that the equation $f(x)=k$, where k is a constant, has exactly two roots.
This means if we draw a horizontal line ($y=k$ is just a horizontal line) it should touch the graph in exactly 2 places
At $y=21$ the line will only cross the graph once
And up until the y intercept the line will cross the graph twice (at 2 different points)
Let's find the y value where $x=0$
When $x=0$: $y=\ f(x)=21-2|2-0|=21-4=17$
At $y=17$ it the line will cross the graph twice
Below where $y=17$ the line only crosses the graph once
We can do this between $17\leq y<21$



$$f(x)=k$$

Vertex of our graph is at (2,21)
$y=af(x-b)$ means we have moved b units to the right and multiplied y by a
The vertex becomes (6,3)
So, we have moved 4 units to the right hence $b=4$ and divided y by 7 hence $a=\frac{1}{7}$

$$a=\frac{1}{7}, b=4$$

Geometry Style Hardest

Range of Possible Values (With Gradient)

A Level October 2020 Pure 2 Q11
At least one solution

$$y=2|x+4|-5.$$

The vertex of the graph is at the point P.

i. Find the coordinates of P

ii. Solve the equation $3x+40=2|x+4|-5$

A line l has equation $y=ax$, where a is a constant. Given that l intersects $y=2|x+4|-5$ **at least once**

iii. Find the range of possible values of a, writing your answer in set notation

i.

$$P(-4,-5)$$

ii.

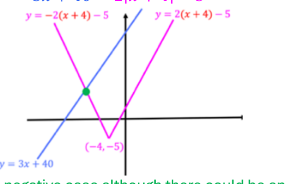
$$3x+40=2|x+4|-5$$

Remember that we take cases for the modulus (plus and minus case)

We solve each case

Positive case:	Negative case:
$3x+40=2(x+4)-5$ $3x+40=2x+8-5$ $x=-37$	$3x+40=2(-x-4)-5$ $3x+40=-2x-8-5$ $5x=-53$ $x=-10.6$
If you look at the graph below you can see this is not possible since the positive case is only for x values bigger than -4 (or just plug this into the question $3x+40=2\|x+4\|-5$ and see if it makes the question true) $3(-37)+40=2\|-37+4\|-5$ $-37\neq 61$	

$$3x+40=2|x+4|-5$$

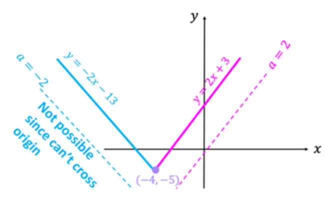


we can see the only solution is the negative case although there could be an intersection far right an that's why the maths don't above was also important

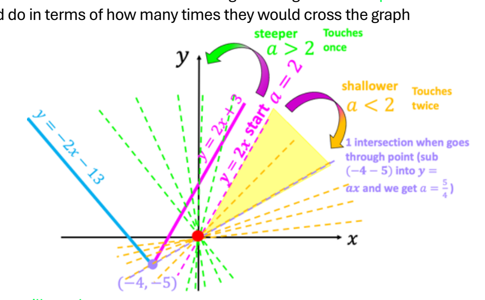
$$x=-10.6$$

iii.

Way 1: Geometric (quicker)
First of all realise the line has to go through the origin
First consider the opposite which is no solutions
a is the gradient of the line
A dashed line parallel line to the pink line would have no solutions (not cross any of the graph)
A dashed line parallel line to the blue line is not possible since would not go through the origin. Also we don't even need to look at the blue case since we can use the pink and swing it round enough to cover all ranges of slopes.



From the pink line let's consider what lines through the origin with steeper and less steeper (shallower) gradients would do in terms of how many times they would cross the graph



Steeper gradients will touch once
Shallower gradients will touch twice, but **only AFTER** they touch/reach the vertex
We can see the **lines in the yellow region don't touch the graph** but ones out of the yellow region i.e. with a gradient more than 2 (the 2 starting from the pink dashed line drawn) and less than $\frac{5}{4}$
So we want the non yellow region

$$\{a: a\leq\frac{5}{4}\cup a>2)$$

Most students hate this so I will show a few algebraic ways

$$2|x+4|-5=ax$$

Way 2: Algebraic (take cases)

Note: I will colour code like usual when graphing, so don't try and link the following to the colours in way 1 above

Positive case:	Negative case
$x+4>0, x\geq -4$ Hence, we get $2(x+4)-5=ax$ $2x+8-5=ax$ $x=\frac{3}{a-2}$ But we need to consider the restriction on x, so we can say $\frac{3}{a-2}\geq -4$ Solving a rational inequality now, have to be careful as don't know the sign of the denom and might swpa the inequality sign. Either take cases or multiply both sides by $(x-2)^2$ since we know that is positive Let's just take cases here	$x+4<0, x<-4$ Hence, we get $-2(x+4)-5=ax$ $-2x-8-5=ax$ $x=\frac{-13}{a+2}$ But we need to consider the restriction on x, so we can say $\frac{-13}{a+2}<-4$ Solving a rational inequality now, have to be careful as don't know the sign of the denom and might swpa the inequality sign. Either take cases or multiply both sides by$(x-2)^2$ since we know that is positive Lets just take cases here

Denom pos	Denom neg
$a-2>0$ $a>2$ Hence, $3\geq -4(a-2)$ $3\geq -4a+8$ $a\geq\frac{5}{4}$ Also need $a>2$ So, $a>2$	$a-2<0$, $a<2$ Hence, $3\leq -4(a-2)$ $3\leq -4a+8$ $a\leq\frac{5}{4}$ This includes $a<2$ So, $a\leq\frac{5}{4}$

Denom pos	Denom neg
$a+2>0$, $a>-2$ Hence, $-13<-4(a+2)$ $13>4a+8$ $a<\frac{5}{4}$ Also need $a>-2$ So, $-2<a<\frac{5}{4}$	$a+2<0$, $a<-2$ Hence, $-13>-4(a+2)$ $13<4a+8$ $a>\frac{5}{4}$ Also need $a<-2$ So, no solution since not <-2

Take the union of both cases. Hence, overall we have $a\in\left(-\infty,\frac{5}{4}\right]\cup(2,\infty)$

June 2025 Pure Paper 2 Q 12
No solution

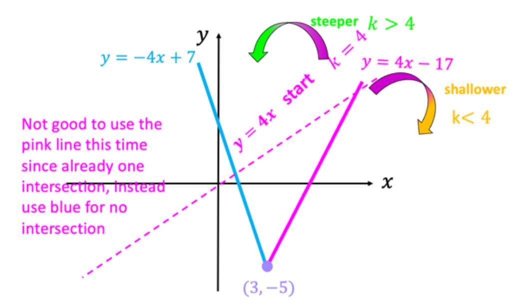
The function g is defined by

$$f(x)=4|x-3|-5, x\in\mathbb{R}$$

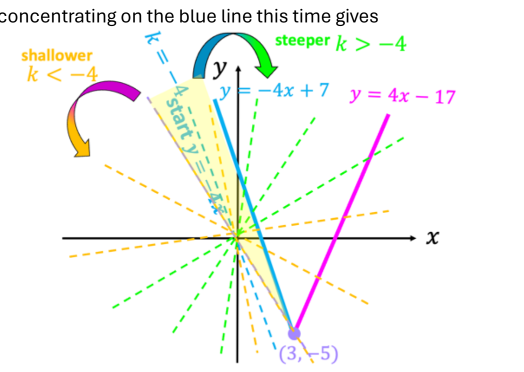
The function h is defined $h(x)=kx, x\in\mathbb{R}$
Where k is a constant
Given that the equation $f(x)=h(x)$ has **no solutions**
Find the range of values of k

Way 1:Geometric

vertex at $(3,-5)$



So concentrating on the blue line this time gives



Plug in the vertex for when the line crosses the vertex

$$-5=k(3)$$
$$k=-\frac{5}{3}$$

From the pink blue let's consider what lines through the origin with steeper and less steeper (shallower) gradients would do in terms of how many times they would cross the graph

Steeper gradients will touch
Shallower gradients will touch but **only AFTER** they touch/reach the vertex, they **wont touch before** they reach the vertex
We can see the **lines in the yellow region don't touch** which is the region we want

$$-4\leq k<-\frac{5}{3}$$

Way 2: Algebraic (take cases)
For algebraic way do as on the left, but then take the opposite region since want no solutions. The general method is

- 1 or 2 solutions: Find union of both cases
- 2 solutions: Find intersection of both cases
- 1 solution: symmetric difference of both cases
- no solution: opposite region of whatever the union was

Way 3: Algebraic (use discriminant)

$$4|x-3|-5=kx$$
$$4|x-3|=kx+5$$
$$\left(4(x-3)\right)^2=(kx+5)^2$$
$$16(x^2-6x+9)=k^2x^2+10kx+25$$
$$16x^2-96x+144=k^2x^2+10kx+25$$
$$(16-k^2)x^2+(-96-10k)x+119=0$$

We want no solutions hence $b^2-4ac=0$

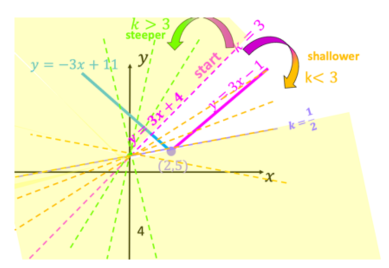
$$(-96-10k)^2-4(16-k^2)(119)=0$$
$$576k^2+1920k+1600=0$$
$$k=-\frac{5}{3},$$

This doesn't help with our lower bound though

June 2024 Paper 1 Q6
Two solutions

A straight line l has equation $y=kx+4$, where k is a constant. Given that l intersects $y=3|x-2|+5$ at **two distinct points**, find the range of possible values of k.

Way 1: Geometric
Here we need to touch twice. We know the line $y=kx+4$ has a y intercept of 4
Like before,
First of all realise the line has to go through a y intercept of 4
First consider the opposite which is no solutions
k is the gradient of the line



Firstly, plug the vertex $(2,5)$ into $y=kx+4$ since intersects to find a

$$5=2k+4$$
$$k=\frac{1}{2}$$

Stepper gradients will touch once (which is not good since we want twice)
Shallower gradients will touch TWICE, but **only BEFORE** they touch/reach the vertex
We can see the **lines in the yellow region don't touch the graph at all** but ones out of the yellow region i.e. with a gradient less than 3 (the 3 starting from the pink dashed line drawn) and more than $\frac{1}{2}$

$$\frac{1}{2}<k<3$$

Way 2: Algebraic

$x-2\geq 0, x\geq 2$	$x-2<0, x<2$
Hence, we get $3(x-2)+5=kx+4$ $3x-6+5=kx+4$ $x=\frac{5}{3-k}$ So, we can say $\frac{5}{3-k}\geq 2$ Now, we have two more cases for the denominator.	Hence, we get $3(2-x)+5=kx+4$ $6-3x+5=kx+4$ $x=\frac{7}{3+k}$ So, we can say $\frac{7}{3+k}<2$ Now, we have two more cases for the denominator.

$3-k>0, k<3$	$3-k<0, k>3$
Hence, $5\geq 6-2k$ $k\geq\frac{1}{2}$ So, $\frac{1}{2}\leq k<3$	Hence, $5\leq 6-2k$ $k\leq\frac{1}{2}$ So, no solution

Hence,

$$\frac{1}{2}\leq k<3$$

$3+k>0, k>-3$	$3+k<0, k<-3$
Hence, $7<6+2k$ $k>\frac{1}{2}$ So, $k>\frac{1}{2}$	Hence, $7>6+2k$ $k<\frac{1}{2}$ So, $k<-3$

Hence,

$$k<-3,\quad k>\frac{1}{2}$$

We want two solutions not one, so we need the overall the intersection which is $\frac{1}{2}<k<3$

Mock set 2 Pure Paper 1 Q 10
No solution

i. Solve the equation $3x+1=|2x-9|+3$

A straight line l has equation $y=kx+1$, where k is a constant
Given that l **does not meet or intersect** the graph with equation

$$y=|2x-9|+3$$

ii. Find the range of possible values of k

i.

$$3x+1=|2x-9|+3$$

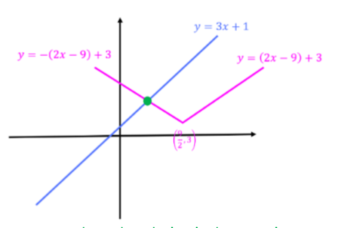
Remember that we take cases for the modulus (plus and minus case)

$$3x+1=(2x-9)+3 \text{ and } 3x+1=[-(2x-9)]+3$$

We solve each case

Positive case:	Negative case:
$3x+1=(2x-9)+3$	$3x+1=[-(2x-9)]+3$
$3x+1=2x-9+3$	$3x+1=-2x+9+3$
$x=-7$	$5x=13$
If you look at the graph below you can see this is not possible since the positive case is only for x values bigger than $\frac{9}{2}$	$x=\frac{13}{5}$

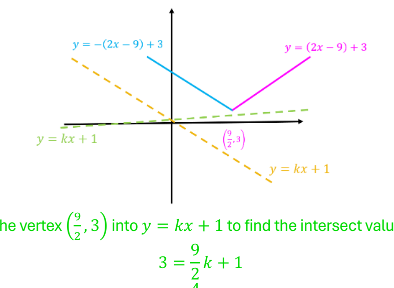
$$3x+1=|2x-9|+3$$



we can see the only solution is the negative case

$$x=\frac{13}{5}$$

ii.



We plug the vertex $\left(\frac{9}{2},3\right)$ into $y=kx+1$ to find the intersect value to find k.

$$3=\frac{9}{2}k+1$$
$$k=\frac{4}{9}$$

$\frac{4}{9}$ is the gradient and we want the line to be less steep hence $k<\frac{4}{9}$.

$$k=-2, k<\frac{4}{9}$$

Modelling

2023 June Paper 2 Q 12

The number of subscribers to two different music streaming companies is being monitored.

The number of subscribers, N_A, in thousands, to company A is modelled by the equation

$$N_A=|t-3|+4 \qquad t\geq 0$$

where t is the time in years since monitoring began.
The number of subscribers, N_N, in thousands, to **company B** is modelled by the equation

$$N_B=8-|2t-6| \qquad t\geq 0$$

where t is the time in years since monitoring began.
Figure 2 shows a sketch of the graph of N_A and the graph of N_B over a 5-year period.

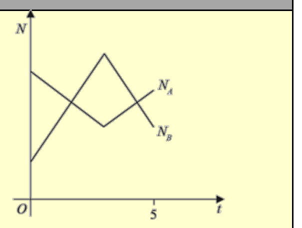


Figure 2

Use the equations of the models to answer all parts below

i. Find the initial difference between the number of subscribers to **company A** and the number of subscribers to **company B.**
When $t=T$ **company A** reduced its subscription prices and the number of subscribers increased.
ii. Suggest a value for T, giving a reason for your answer
iii. Find the range of values of t for which $N_A>N_B$ giving your answer in set notation
iv. State a limitation of the model used for **company**

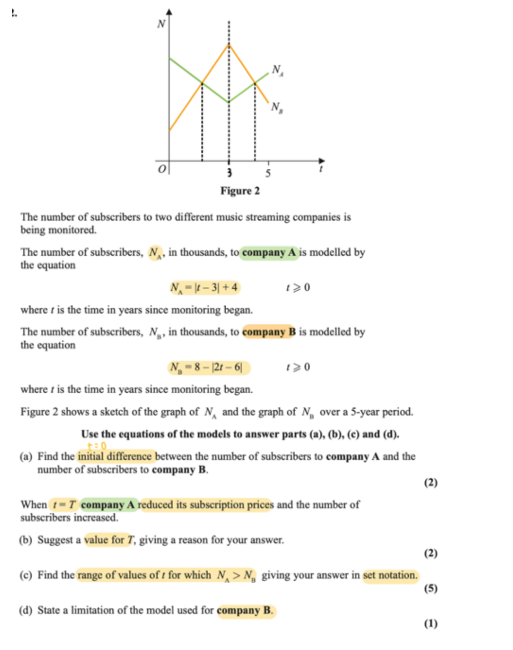


Figure 2

The number of subscribers to two different music streaming companies is being monitored.

The number of subscribers, N_A, in thousands, to company A is modelled by the equation

$N_A=|t-3|+4 \qquad t\geq 0$

where t is the time in years since monitoring began.

The number of subscribers, N_B, in thousands, to company B is modelled by the equation

$N_B=8-|2t-6| \qquad t\geq 0$

where t is the time in years since monitoring began.

Figure 2 shows a sketch of the graph of N_A and the graph of N_B over a 5-year period.

Use the equations of the models to answer parts (a), (b), (c) and (d).

(a) Find the initial difference between the number of subscribers to company A and the number of subscribers to company B. (2)

When $t=T$ company A reduced its subscription prices and the number of subscribers increased.

(b) Suggest a value for T, giving a reason for your answer. (2)

(c) Find the range of values of t for which $N_A>N_B$ giving your answer in set notation. (5)

(d) State a limitation of the model used for company B. (1)

(a) We need to find N_A and N_B at $t=0$:
$N_A=|0-3|+4$
$=3+4$
$=7$ thousand – remember N_A and N_B are given in thousands!
$N_B=8-|2(0)-6|$
$=8-6$
$=2$ thousand M1
the difference is ∴ $7-2=$ 5000 subscribers A1

(b) We need the point when the subscribers of A started increasing again, which is the point when A had the minimum number of subscribers. B1
This occurs at $T=3$ B1

(c) We need to find the intersection points of N_A and N_B.
Equate N_A and N_B:
★ While N_A is decreasing, see that N_B is increasing and vice versa. Hence we need to always "flip" the brackets depending on what sign the t's have (+ or -)

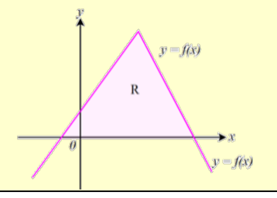
Case I $(t<3)$	Case II $(t>3)$
$\|t-3\|+4=8-\|2t+6\|$	$\|t-3\|+4=8-\|2t-6\|$
N_A is decreasing ∴ t_A must have negative coefficient.	N_B is decreasing ∴ t_B must have negative coefficient.
N_B is increasing ∴ t_A must have positive coefficient.	N_A is increasing ∴ t_A must have positive coefficient.
$(3-t)+4=8-(6-2t)$ flip so $-2t$ and t is positive!	$(t-3)+4=8-(2t-6)$ ☆ don't flip either bracket
flip so t is negative! $3-t+4=8+2t-6$	$t+1=14-2t$ B1
$7-t=2+2t$	$3t=13$
$5=3t$	$t=\frac{13}{3}$
$t=\frac{5}{3}$ M1A1	

So for $N_A>N_B$ to be true, $t<\frac{5}{3}$ or $t>\frac{13}{3}$ A1
We have to give this in set notation
$\{t\in\mathbb{R}\mid t<\frac{5}{3}\}\cup\{t\in\mathbb{R}\mid t>\frac{13}{3}\}$ A1

(d) When $t>7$, the number of subscribers would become negative, which is unrealistic B1

With Integration

The sketch above shows a graph with equation $y=f(x)$ where $f(x)=7-|3x-5|$.
The finite region R is bounded by the graph with equation $y=f(x)$ and the x axis.
Find the area of R, giving your answer in simplest form



R is a triangle:
The base is the distance between the x-axis intercepts

$$7-|3x-5|=0$$
$$7=|3x-5|$$

$7=(3x-5)$ $7+5=3x$ $12=3x$ $x=4$	$7=-(3x-5)$ $7=-3x+5$ $7-5=-3x$ $2=-3x$ $x=-\frac{2}{3}$

$$\text{Distance}=4--\frac{2}{3}=\frac{14}{3}$$

The greatest height of the triangle appears when $|3x-5|=0$, so the greatest height will be 7

$$\text{Area of a triangle}=\frac{1}{2}\text{base}\times\text{height}$$
$$=\frac{1}{2}\times\frac{14}{3}\times 7=\frac{49}{3}$$

$$\int_2^{11}|x-9|dx$$

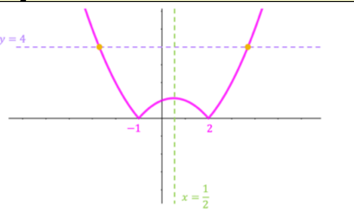
Recall that the modulus function changes based on the values of x

$$f(x)=\begin{cases}x-9, & x\geq 9\\ -x+9, & x\leq 9\end{cases}$$
$$\int_2^{11}|x-9|\,dx=\int_2^9(-x+9)\ dx+\int_9^{11}(x-9)\ dx$$
$$\int_9^{11}(x-9)\ dx=\left[\frac{1}{2}x^2-9x\right]_9^{11}=\left(\frac{1}{2}(11)^2-9(11)\right)-\left(\frac{1}{2}(9)^2-9(9)\right)=2$$
$$\int_2^9(-x+9)\ dx=\left[-\frac{1}{2}x^2+9x\right]_2^9=\left(-\frac{1}{2}(9)^2+9(9)\right)-\left(\frac{1}{2}(2)^2-9(2)\right)=\frac{49}{2}$$

Hence combine these two values we calculated:

$$2+\frac{49}{2}=\frac{53}{2}$$

Sketch the curve $y=|x^2-x-2|$ and find the area within the following boundaries:
The curve $y=|x^2-x-2|$ and the vertical lines $x=\frac{1}{2}$ and the horizontal line $y=4$ and the positive x axis



Step 1: Find the coordinates of where the line $y=4$ crosses the curve

$$x^2-x-2=4$$
$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$x=3\ \ \&\ \ x=-2$$

We are only concerned with the positive x axis hence we only care about the coordinate (3,4)

Step 2: We have to split this up into two integrals, with two different scenarios

$$f(x)=|x^2-x-2|$$

We are concerned with the positive x axis, hence we will split our limits from:

$$x=\frac{1}{2}\text{ to }x=2\ \ \&\ \ x=2\text{ to }x=3$$
$$Area=\int_{\frac{1}{2}}^2[4-(-x^2+x+2)]\,dx+\int_2^3[4-(x^2-x-2)]\,dx$$

Simplify:

$$Area=\int_{\frac{1}{2}}^2[x^2-x+2]\,dx+\int_2^3[-x^2+x+6]\,dx$$

Step 3: These are very elementary integrals hence continue as such:

$$\int_2^3[-x^2+x+6]\,dx=\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+6x\right]_2^3=\left(-\frac{1}{3}(3)^3+\frac{1}{2}(3)^2+6(3)\right)-\left(-\frac{1}{3}(2)^3+\frac{1}{2}(2)^2+6(2)\right)=\frac{13}{6}$$
$$\int_{\frac{1}{2}}^2[x^2-x+2]\,dx=\left[\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{\frac{1}{2}}^2=\left(\frac{1}{3}(2)^3-\frac{1}{2}(2)^2+2(2)\right)-\left(\frac{1}{3}\left(\frac{1}{2}\right)^3-\frac{1}{2}\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)\right)=\frac{15}{4}$$

Hence combine these two values we calculated:

$$\frac{13}{6}+\frac{15}{4}=\frac{71}{12}$$

Domain and range

See relevant sheet

Hardest Solving Types Ever (very unlikely to come up)

$$|2x+1|+|2x-1|=3$$

Re-arrange to get one modulus on one side

$$|2x+1|=3-|2x-1|$$

Square both sides

$$(2x+1)^2=(3-|2x-1|)^2$$
$$4x^2+4x+1=9-6|2x-1|+(2x-1)^2$$
$$4x^2+4x+1=9-6|2x-1|+4x^2-4x+1$$
$$9-8x=|12x-6|$$

Positive case	Negative case
$9-8x=12x-6$ $x=\frac{3}{4}$	$9-8x=-(12x-6)$ $x=-\frac{3}{4}$

Check:

$$\left|2\left(\frac{3}{4}\right)+1\right|+\left|2\left(\frac{3}{4}\right)-1\right|=\frac{5}{2}+\frac{1}{2}=3$$
$$\left|2\left(-\frac{3}{4}\right)+1\right|+\left|2\left(-\frac{3}{4}\right)-1\right|=\frac{1}{2}+\frac{5}{2}=3$$

Both solutions are valid

To Try:

$x<4-|2x+1|$
$|x|+3\geq|4x-1|$
$|x|+|x-2|>5$
$|x+1|+|6-2x|\leq 10$

Answers
$-5<x<1$
$-\frac{2}{3}\leq x\leq\frac{4}{3}$
$x<-\frac{3}{2}, x>\frac{7}{2}$
$-\frac{5}{3}\leq x\leq 5$